Hamiltonian hierarchy and the Hulthén potential

B. Gönül, O. Özer, Y. Cançelik ,and M. Koçak
Department of Engineering Physics, University of Gaziantep,
27310 Gaziantep-Türkiye

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Abstract

We deal with the Hamiltonian hierarchy problem of the Hulthén potential within the frame of the supersymmetric quantum mechanics and find that the associated superymmetric partner potentials simulate the effect of the centrifugal barrier. Incorporating the supersymmetric solutions and using the first-order perturbation theory we obtain an expression for the energy levels of theHulthén potential which gives satisfactory values for the non-zero angular momentum states.

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1 Introduction

TheHulthén potential [1,2] is one of the important short-range potentials in physics. The potential is given by

$$V(r) = -\frac{Ze^2\delta e^{-\delta r}}{(1-e^{-\delta r})} \tag{1}$$

where Z is a constant and δ is the screening parameter. If the potential is used for atoms, the Z is identified with the atomic number. This potential is a special case of the Eckart potential [3] which has been widely used in several branches of physics and its bound-state and scattering properties have been investigated by a variety of techniques (see e.g., [4] and references therein).

Unfortunately, the radial Schrödinger equation for the Hulthén potential can be solved analytically only for the states with zero angular momentum [1,2,5,6]. For $\ell \neq 0$, a number of methods have been employed to evaluate bound-state energies numerically [4,7-17]. In connection with this, we present in this letter a method within the frame of supersymmetric quantum mechanics (SUSYQM) using an effective Hulthén potential for non-zero angular momentum states, which can be solved analytically.

In supersymmetric quantum mechanics (for a recent review see [18]) one often deals with hierarchy problem. Within the context of the SUSYQM one can generate a Hamiltonian hierarchy, the adjacent members of which are supersymmetric partners in that they share the same eigenvalue spectrum except for the missing ground state. In the case of Coulomb potential $V_c(r)$, the Hamiltonian hierarchy corresponds to the addition of an appropriate centrifugal potential and the so-called accidental degenarcy is recovered as a natural consequence [19]. In this letter

we shall examine the implication of the Hamiltonian hierarchy for the Hulthén potential. At small values of the radial coordinate r, the Hulthén potential behaves like a Coulomb potential whereas for large values of r it decreases exponentially so that its capacity for bound state is smaller then $V_c(r)$. In contrast to the Hulthén potential, the Coulomb problem is analytically solvable for all energies and all angular momenta. Because of the similarity and points of contrast mentioned above between Coulomb and Hulthén potentials, it may be of considerable interest to generate the supersymmetric partners of the Hulthn potential and study their eigenfunctions and eigenvalues. In the following we outline the basic idea of the SUSYQM and set up the formalism for dealing with the Hulthén problem.

2 Supersymmetric solution of theHulthén potential

The supersymmetric formalism has already been used to study some aspects of the Hulthén potential [15,17]. Here the exact analytical solution for this potential is re-obtained for $\ell = 0$ state in the light of the works described in Refs. [15-17,20] to show the consistency of the method and see how partners of the Hulthén potential simulate the effect of the centrifugal barrier, which makes clear the physics behind partner Hamiltonians in connection with the states having $\ell \neq 0$ angular momenta.

In the approach followed here the first step taken is to look for an effective potential similar to the original Hulthén potential. Inspired by SUSYQM, we propose an ansatz for the superpotential,

$$W_{\ell+1}(r) = -\frac{\hbar}{\sqrt{2m}}\frac{(\ell+1)\delta e^{-\delta r}}{(1-e^{-}\delta r)} + \sqrt{\frac{m}{2}}\frac{e^2}{\hbar}\left[\frac{1}{(\ell+1)} - \frac{(\ell+1)\beta}{2}\right] \tag{2}$$

where $(\ell+1)$ denotes the partner number with $\ell = 0, 1, 2, ...$, and $\beta = \frac{\hbar^2\delta}{me^2}$ which is a dimensionless quantity. This kind of superpotential choice leads to the $(\ell+1)$-th member of the Hamiltonian hierarchy:

$$V_{\ell+1}(r) - E^{n=0}_{\ell+1} = W^2_{(\ell+1)}(r) - \frac{\hbar}{\sqrt{2m}}\frac{d}{dr}W_{(\ell+1)}(r)\ , \tag{3}$$

$$V_{\ell+1}(r) = \frac{\hbar^2}{2m}\frac{\ell(\ell+1)\delta^2 e^{-2\delta r}}{(1-e^{-\delta r})^2} - e^2\frac{\delta e^{-\delta r}}{(1-e^{-\delta r})}\left[1 - \ell(\ell+1)\frac{\beta}{2}\right] \tag{4}$$

We introduce an expression for the bound-state energies of the above potential, considering the shape invariance requirement [21],

$$E^n_{\ell+1} = -\frac{me^4}{2\hbar^2}\left[\frac{1}{(n+\ell+1)} - \frac{(n+\ell+1)(\beta)}{2}\right]^2 \quad ; \quad n = 0, 1, 2, ... \tag{5}$$

and the corresponding ground-state eigenfunctions are

$$\Psi^{n=0}_{(\ell+1)}(r) = N\ exp\left(-\frac{\sqrt{2m}}{\hbar}\int^r W_{(\ell+1)}(r')dr'\right)$$

$$= N\ (1-e^{-\delta r})^{\ell+1} exp\left\{-\frac{me^2}{\hbar^2}\left[\frac{1}{(\ell+1)} - \frac{(\ell+1)\beta}{2}\right]r\right\} \tag{6}$$

It is reminded that for a number of purposes it is convenient to have the wavefunction in such a compact analytical form. The first eigenfunction corresponds to the minimum energy for each ℓ. In terms of the hierarchy of Hamiltonians, we present here the lowest state wavefunctions for each member. The excited state wavefunctions can be determined [18,26] from the usual approach in SUSYQM.

For $\ell = 0$ the potential in Eq. (4) leads to the usual Hulthén potential which has an interesting property such that when the angular momentum is zero it is not shape invariant in the sense expressed in [21]. However, it is still possible to construct a general form of the potentials in the shape invariant super-family of Hamiltonians as seen in Eq. (4) where the first member corresponds to the Hulthén potential which can be solved exactly in analytic form. One can easily verify that the energy eigenvalues and eigenfunctions for $\ell = 0$ case of Eqs. (5) and (6) are the same given in Refs. [5,6]. This supports the suggestion [17,22-24] that the Gedenshtein's condition of shape invariance is sufficient but not a necessary condition in the construction of exactly solvable but non-shape invariant potentials.

Eq.(4) can be rearranged as

$$V_H^{eff}(r) = V_{\ell+1}(r) = -e^2\frac{\delta e^{-\delta r}}{(1-e^{-\delta r})} + \frac{\ell(\ell+1)\hbar^2\delta^2}{2m(1-e^{-\delta r)^2}}e^{-\delta r} \tag{7}$$

which is known in literature as the approximate Hulthén effective potential introduced by Greene and Aldrich [25] in their method to generate pseudo-Hulthen wave functions for $\ell \neq 0$ states. For small δr, Eq. (7) is a good approximation to the realisticHulthén effective potential, and unlike the original case the radial Schrödinger equation for this potential is solvable analytically through Eqs. (5) and (6). In addition, the partner potentials in Eq. (7) gives the necessary repulsive core due to angular momentum. For instance, for small r the second term in Eq. (7) behaves as a p-wave centrifugal barrier for the second member of the super-family. Since we know that the centrifigal potential is effective only in this region (i.e., small r), eigensolution of the potential for $\ell = 1$ in Eq. (6) can be regarded as the approximate p-wave solution for the Hulthen potential. Clearly, one can get other supersymmetric partners and their solutions in an explicit form for $\ell \neq 0$ states. The present simple and elegant method is a clear cut of the iteration technique introduced by Laha *et al.* [15,16]

For the sake of completeness, it is of interest to note that for small values of δ, the potential in Eq. (7) closely approximates the effective Coulomb potential rather well,

$$V_H^{eff}(r, \delta \cong 0) \rightarrow V_C^{eff}(r) = -\frac{e^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \tag{8}$$

and the corresponding energy eigenvalue for the potential of Eq. (8), together with its ground state wavefunction for $\ell \neq 0$ states, obtained easily via Eqs. (5) and (6) overlap with those, e.g. in Ref. [26]. This makes clear the work of Lam and Varshni [5] in which they showed that if one uses as trial functions eigenvectors of theHulthén potential rather than those of the simple Coulomb potential, excellent results for the energies of the states of the screened Coulomb potential can be obtained with simple variational wave functions containing only one parameter.

An important quantity of interest for the Hulthén potential (and for other similar screened potentials) is the critical screening parameter δ_c, which is that value of δ for which the binding energy of the level in question becomes zero. Using Eq. (5), in atomic units,

$$\delta_c = \frac{2}{(n+\ell+1)^2} \tag{9}$$

which works well for all n values in case $\ell = 0$ when compared to those in Table III of Ref. [4], but fails for non-zero angular momentum states. Consequently, the eigenenergies obtained via Eq. (5) for $\ell \neq 0$ states deviates from the accurate values obtained by numerical techniques and presented in Table I of Ref. [4]. This may be understood as follow. If Eq. (7) is written in the form

$$V_H^{eff}(r) = -e^2\frac{\delta e^{-\delta r}}{(1-e^{-\delta r})} + \frac{\ell(\ell+1)(\hbar^2)}{2mr^2} + \left[\frac{\ell(\ell+1)\hbar^2\delta^2}{2m(1-e^{-\delta r})^2}e^{-\delta r} - \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right] \tag{10}$$

the exact energy eigenvalues for the realistic effective Hulthén potential may be given as

$$E_H^{n\ell} = -\frac{me^4}{2\hbar^2}\left[\frac{1}{(n+\ell+1)} - \frac{(n+\ell+1)\beta}{2}\right]^2 + \Delta E \tag{11}$$

where ΔE is the contribution, which does not appear in Eq. (9), due to the last term in Eq. (10). The clear interpretation of Eqs. (10) and (11) is that the potential barrier term prevents us to build the super-family as in the $\ell = 0$ case, since the potential-the first two terms in Eq. (10)-is not exactly solvable hence the supersymmetry is broken for $\ell \neq 0$ due to the potential barrier term. It is easy however to verify that for small values of δ, ΔE goes to zero while Eq. (10) becomes an expression for the effective Coulomb potential in which case the accidental degenarcy is recovered as a natural consequence.

The usefulness of theHulthén potential would be enhanced if one obtains an analytical expression for the exact energies of the non-zero angular momentum states. The work along this line is in progress in the frame of broken supersymetry. Further, in the light of the supersymmetric solutions discussed in this letter we suggest here, as an alternative to other various methods [4,7-14] investigating the bound-state properties of theHulthén potential, an elegant approach for the calculation of the whole energy spectrum of the potential using the first-order perturbation theory,

$$E_H^{n\ell} = E_{\ell+1}^n + \frac{\ell(\ell+1)\hbar^2}{2m}\int_0^\infty \left[\psi_{\ell+1}^n(r)\right]^2\left(\frac{1}{r^2} - \frac{\delta^2}{(1-e^{-\delta r})^2}e^{-\delta r}\right)dr \tag{12}$$

which gives satisfactory values when compared (see Table 1) with the results obtained by the various methods for the eigenenergies of $\ell \neq 0$ levels. The accuracy of the present calculations may be improved incorparating higher-order perturbations for in particular large values of the screening parameter.

3 Conclusion

We have obtained the exact analytical eigenfuntions and eigenvalues for the Hulthén potential within the framework of SUSYQM for the case $\ell = 0$. The approach consists of making an ansatz in the superpotential which satisfies the Riccati equation by an effective potential. For $\ell = 0$ the effective potential obtained is identical to the Hulthén potential. However, for $\ell \neq 0$ the effective supersymmetric potential has a slightly different structure than the Hulthén potential. This deviation has led us to introduce a simple expression that yields reasonable results for the non-zero angular momentum state energies. We stress that even though the problem has been attacked by different methods our simple and elegant methodology is powerful because it provides an insight into the relation between theoretical partner Hamiltonians in the

frame of supersymmetric quantum mechanics and physical states of the system considered. We hope to stimulate further examples of applications of SUSYQM in important problems of nuclear and atomic physics.

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Table 1: Energy eigenvalues of the Hulthén potential as a function of screening parameter for various states in atomic units.

State	δ	Present Calculations	Numerical Integration (Ref. [4])	Variational (Ref. [4])	Lai and Lin (Ref. [7])	Patil (Ref. [9])	Tang and Chan (Ref. [12])	Matthys and De Meyer (Ref. [14])
2p	0.025	0.1127605	0.1127605	0.1127605		0.11276		0.1127604
	0.050	0.1010425	0.1010425	0.1010425	0.101043	0.10104	0.1010424	0.1010425
	0.075	0.0898478	0.0898478	0.0898478		0.08985		
	0.100	0.0791794	0.0791794	0.0791794	0.079179	0.07918	0.0791794	0.0791794
	0.150	0.0594415	0.0594415	0.0594415		0.059445		0.0594415
	0.200	0.0418854	0.0418860	0.0418860	0.041886	0.041895	0.0418857	0.0418860
	0.250	0.0266060	0.0266111	0.0266108				
	0.300	0.0137596	0.0137900	0.0137878	0.013790			0.0137900
	0.350	0.0036146	0.0037931	0.0037734	0.003779	0.038375		
3p	0.025	0.0437068	0.0437069	0.0437069	0.043707	0.0437085		0.0437071
	0.050	0.0331632	0.0331645	0.0331645	0.033165	0.033185	0.03316518	0.0331650
	0.075	0.0239331	0.0239397	0.0239397		0.0240165		
	0.100	0.0160326	0.0160537	0.0160537	0.016054	0.01622	0.01606772	0.0160537
	0.150	0.0043599	0.0044663	0.0044660	0.004466	0.046995		0.0044664
3d	0.025	0.0436030	0.0436030	0.0436030	0.043603	0.0436025		0.0436030
	0.050	0.0327532	0.0327532	0.0327532	0.032753	0.032745	0.0327532	0.0327532
	0.075	0.0230306	0.0230307	0.0230307		0.02299		
	0.100	0.00144832	0.0144842	0.0144832	0.014484	0.01439	0.0144842	0.0144842
	0.150	0.0132820	0.0013966	0.0013894	0.001391	0.0013755		0.0013965
4p	0.025	0.0199480	0.0199489	0.0199489	0.019949	0.01995		0.0199490
	0.050	0.0110430	0.0110582	0.0110582	0.011058	0.011075	0.0110725	0.0110583
	0.075	0.0045385	0.0046219	0.0046219	0.004622	0.0046585		0.0046224
	0.100	0.0004434	0.0007550	0.0007532	0.000754	0.000752		
4d	0.025	0.0198460	0.0198462	0.0198462	0.019846	0.019845		0.0198462
	0.050	0.0106609	0.0106674	0.0106674	0.010667	0.01068	0.0106690	0.0106674
	0.075	0.0037916	0.0038345	0.0038344	0.003834	0.003875		0.0038346
4f	0.025	0.0196911	0.0196911	0.0196911	0.019691	0.01969		0.0196911
	0.050	0.0100618	0.0100620	0.0100620	0.010062	0.010045	0.0100620	0.0100619
	0.075	0.0025468	0.0025563	0.0025557	0.002556	0.002557		0.0025563

Table 2: Continue of Table 1.

State	δ	Present Calculations	Numerical Integration (Ref. [4])	Variational (Ref. [4])	Lai and Lin (Ref. [7])	Patil (Ref. [9])	Tang and Chan (Ref. [12])	Matthys and De Meyer (Ref. [14])
5p	0.025	0.0094011	0.0094036				0.0094087	
	0.050	0.0026056	0.0026490					
5d	0.025	0.0092977	0.0093037				0.0093050	
	0.050	0.0022044	0.0023131					
5f	0.025	0.0091507	0.0091521				0.0091523	
	0.050	0.0017421	0.0017835					
5g	0.025	0.0089465	0.0089465				0.0089465	
	0.050	0.0010664	0.0010159					
6p	0.025	0.0041493	0.0041548					
6d	0.025	0.0040452	0.0040606					
6f	0.025	0.0038901	0.0039168					
6g	0.025	0.0036943	0.0037201					